CONIC SECTION

Conic or Conic Section:

It is defined as the locus of a point P which movies in such a way that its distance from a fixed point S always bears a constant ratio to its distance from a fixed line, all being in the same plane.

SOME IMPORTANT DEFINITIONS

Focus: The fixed point (S) is called the focus of the conic section.

Directrix: The fixed straight line is called the directrix of the conic section. In general, every conic has four foci, two of them are real and the other two are imaginary.
Due to two real foci, every conic has two directrices corresponding to each real focus.

Eccentricity: The constant ratio is called the eccentricity of the conic section. It is denoted by **e**.

Axis: The straight line passing through the focus and perpendicular to directrix is called the axis of the conic section.

Vertex: The points of intersection of the conic section and the axis are called vertices of the conic section.

Centre: It is the point which bisects every chord of the conic passing through it.

Latus-rectum: The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.

CLASSIFICATION OF THE CONICS

If eccentricity (e)

1. $e < 1 \Rightarrow$ **Ellipse**
2. $e = 1 \Rightarrow$ **Parabola**
3. $e > 1 \Rightarrow$ **Hyperbola**
4. $e = 0 \Rightarrow$ **Circle**

DIFFERENT REFLECTIONS

(i) The reflection of a point (x, y) in

a. X – axis is (x, – y)
b. Y – axis is (– x, y)
c. line y = x is (y, x)
d. line y = –x is (– y, – x)
e. origin is (– x, – y)

Let F(x, y) = 0 be the equation of a curve C, then C is symmetrical about

a. X – axis iff F(x, y) = F(x, – y)
b. Y – axis iff F(x, y) = F(–x, y)
c. line x – y = 0 iff F(x, y) = F(y, x)
d. line x + y = 0 iff F(x, y) = F(– y, – x)
e. origin iff F(x, y) = F(– x, – y)

Note: If a curve is symmetrical about both the axes, then it must be symmetrical abut the origin.

GENERAL PROPERTIES OF A CONIC

Let S = 0, be the equation of a conic, then equation of -

(i) **Chord** joining the points $(x_1,y_1)$and (x_2, y_2) on the conic is $\mathbf{S_1 + S_2 = S_{12}}$

(ii) **Tangent** to the conic at (x_1, y_1) on the conic is $\mathbf{S_{11} = 0}$

(iii) **Chord of Contact** of the point (x_1, y_1) w.r.t. the conic is $\mathbf{S_{11} = 0}$

(iv) **Polar** of the point (x_1, y_1) with respect to the conic S = 0 is $\mathbf{S_1 = 0}$ (where (x_1, y_1) is not the centre.)

(v) **Chord** of the conic with (x_1, y_1) as its mid-point is $\mathbf{S_1 = S_{11}}$

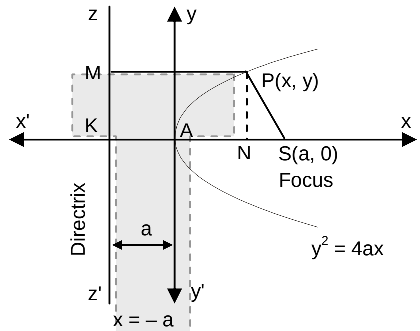
(vi) **Pair of Tangents** from (x_1, y_1) to the conic S = 0 is $\mathbf{SS_1 = S_{11}^2}$

THE PARABOLA

It is defined as the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called the focus and the fixed line is called the directrix of the parabola.

For parabola **eccentricity (e) = 1**



Equation of Parabola:

Let S be the focus, zz' be the directrix and A be the vertex of the parabola which is origin.

Let P (x, y) be any point on the parabola

$\therefore$ By definition SP = PM

$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = x + a$

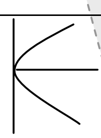
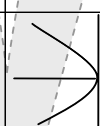
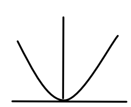
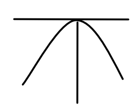
or, $x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$ or, $\mathbf{y^2 = 4ax}$

which is the required equation of the parabola.

Results for the parabola $y^2 = 4ax$.

1. Vertex A (0, 0)
2. Focus S (a, 0)
3. Equation of directrix x + a = 0
4. Equation of axis y = 0 (i.e., x-axis)
5. LL' is the latus – rectum of the parabola and LL' = 4a

DIFFERENT FORMS OF PARABOLA

ITEM	$\mathbf{y^2 = 4ax}$	$\mathbf{y^2 = -4ax}$	$\mathbf{x^2 = 4ay}$	$\mathbf{x^2 = -4ay}$
				
VERTEX	(0, 0)	(0, 0)	(0, 0)	(0, 0)
FOCUS	(a, 0)	(–a, 0)	(0, a)	(0, –a)
AXIS	y = 0	y = 0	x = 0	x = 0
DIRECTRIX	x = –a	x = a	y = –a	y = a
LATUS-RECTUM	4a	4a	4a	4a
PARAMETRIC COORDINATES	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
FOCAL DISTANCE	x + a	x – a	y + a	y – a

GENERAL EQUATION OF PARABOLA

Let S (h, k) be the focus and ax + by + c = 0 be the directrix of the parabola.

Let P(x , y) be any point on the parabola.

∴ Distance between P and S = distance of P from directrix

$$\therefore \sqrt{(x-h)^2+(y-k)^2} = \pm\frac{ax+by+c}{\sqrt{a^2+b^2}}$$

or $(a^2 + b^2)(x^2 + y^2 - 2bx - 2ky + h^2 + k^2) = a^2x^2 + b^2y^2 + c^2 + 2abxy + 2bcy + 2acx$

$\Rightarrow$ **$(bx - ay)^2 + 2gx + 2fy + e = 0$**

A second degree equation represents a parabola if the second degree terms forms a perfect square.

INTERSECTION OF LINE AND PARABOLA

The line y = mx + c & the parabola **$y^2 = 4ax$**

- Intersects in two real and distinct points if **$c <$** $\frac{a}{m}$
- Intersects in two imaginary points if **$c >$** $\frac{a}{m}$
- is a tangent if $c = \frac{a}{m}$

Point Of Contact: $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

EQUATIONS OF TANGENT IN DIFFERENT FORMS

Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is **$y\, y_1 = 2a(x + x_1)$** or **$S_{11} = 0$.**

Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

Slope Form

The equation of a tangent of slope m to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}$$

The coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Point of intersection of tangents at any two points on the parabola

The point of intersection of tangents at two points P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$(at_1t_2,\ a(t_1 + t_2))$$

Remark: x-coordinate of the point of intersection of tangents at P and Q on the parabola is the G.M. of the x-coordinates of P and Q, and y-coordinate is the A.M. of y-coordinates.

EQUATION OF NORMAL

The equation of tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is given by $yy_1 = 2a(x + x_a)$

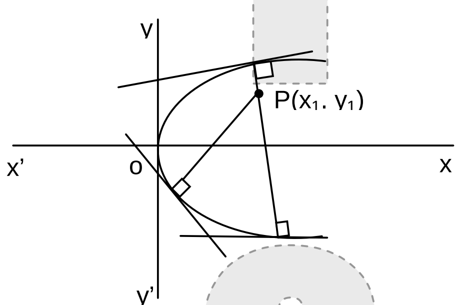
or, $y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$. Slope of tangent $= 2a/y_1$.

$\therefore$ Slope of normal $= -\frac{y_1}{2a}$

$\therefore$ The equation of normal at (x_1, y_1) is $\mathbf{(y - y_1) = -\frac{y_1}{2a}(x - x_1)}$

NUMBER OF NORMALS

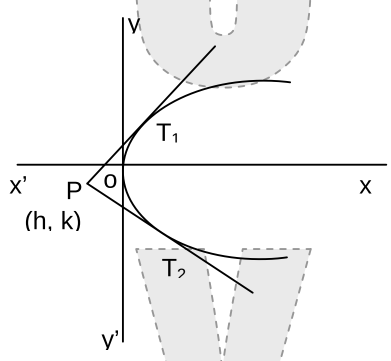
In general 3 normals can be drawn from a point to a parabola.



NUMBER OF TANGENTS

Two tangents can be drawn from a point to a parabola.

The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.



EQUATION OF THE CHORD

Equation of the chord through P(x_1, y_1) and Q(x_2, y_2) is

$$\mathbf{y(y_1 + y_2) = 4ax + y_1 y_2} \quad \text{... (i)}$$

Equation of the chord joining P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$

Replacing y_1 by $2at_1$ and y_2 by $2at_2$ in equation (i) we get

$$\mathbf{y(t_1 + t_2) = 2x + 2at_1 t_2} \quad \text{... (ii)}$$

If the chord PQ passes through the focus (a, 0), then

$$0(t_1 + t_2) = 2a + 2at_1 t_2$$

$$\Rightarrow \mathbf{t_1 t_2 = -1 \text{ or } y_1 y_2 = -4a^2}$$

PROPERTIES OF THE PARABOLA

- The tangents at the extremities of any focal chord of a parabola intersect at right angles at the directrix.
- The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
- The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points
- If m is the slope of the normal to $y^2 = 4$ ax, then its equation is $y = mx - 2am - am^3$.
- Three normals can be drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$.
- The point of intersection of a pair of perpendicular tangents on the directrix.
- The locus of the point of intersection of perpendicular tangents is the directrix.
- If $P(t_1)$ and $Q(t_2)$ are the extremities of a local chord of a parabola $y^2 = 4ax$ then $t_1t_2 = -1$.
- The semi-latus rectum is the harmonic mean between the segments of a focal chord.

Examples

Ex. The equation of the directrix of the parabola $y^2 - 4y + 4x = 0$ is:

(A) $x - 2 = 0$ (B) $x + 2 = 0$ (C) $y - 2 = 0$ (D) $y + 2 = 0$

Sol. The given equation is $y^2 - 4y + 4x = 0$ or $(y - 2)^2 = -4(x - 1)$ ….. (i).

which is a form of $Y^2 = -4a X$, where $Y = y - 2$, $X = x - 1$ and $a = 1$.

$\therefore$ Directrix of the parabola is $X = a \Rightarrow x - 1 = 1$

$\Rightarrow x - 2 = 0$.

$\therefore$ The correct answer is (A).

Ex. The equation of the parabola with focus (– 8, – 2) and directrix $y = 2x - 9$ is:

(A) $x^2 + y^2 + 2xy + 16x + 2y + 253 = 0$ (B) $x^2 + 4y^2 + 4xy + 16x + 2y + 259 = 0$

(C) $4x^2 + y^2 + 4xy + 16x + 2y + 257 = 0$ (D) None of these

Sol. Let P (x, y) be any point on the parabola

$\therefore$ Distance of P from focus (–8, –2) = Distance of P from directrix $2x - y - 9 = 0$.

$$\therefore \sqrt{(x+8)^2 + (y+2)^2} = \frac{2x - y - 9}{\sqrt{2^2 + 1^2}}$$

or $5[(x + 8)^2 + (y + 2)^2] = (2x - y - 9)^2$

or $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$,which is the required equation of parabola.

$\therefore$ Hence (B) is correct.

Ex. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if:

(A) $ln = am^2$ (B) $lm = an^2$ (C) $nm = al^2$ (D) None of these

Sol. Compare the given equation with $y = Mx + a/M$.

The given equation is $lx + my + n = 0$

i.e. $y = -\frac{l}{m}x - \frac{n}{m}$.

Here $M = -\frac{l}{m}$ and $\frac{a}{M} = -\frac{n}{m}$

$\therefore M\frac{a}{M} = \left(-\frac{l}{m}\right)x\left(\frac{-n}{m}\right)$ or $am^2 = ln$.

Hence (A) is correct.

THE ELLIPSE

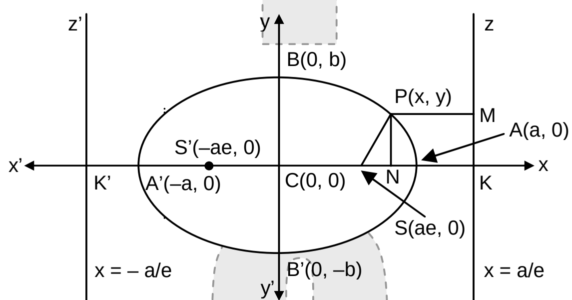
ELLIPSE: The ellipse is a locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point is called the focus and the fixed line is called the directrix.

Hence, If S be focus of the ellipse, P any point on it and PM is perpendicular distance of the directrix from P, then $\frac{SP}{PM}$ = e or, SP = e. PM (where e < 1)

For ellipse **eccentricity (e) < 1**

EQUATION OF ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Where $b^2 = a^2(1-e^2)$. (b < a) is the standard equation of ellipse.



GENERAL EQUATION OF ELLIPSE The equation of the ellipse whose focus is the point (h, k), the directrix is the line Ax + By + C = 0, and whose eccentricity is e,

is $\mathbf{(x-h)^2 + (y-k)^2 = e^2} \,.\, \frac{(Ax+By+C)^2}{A^2+B^2}$

SUMMARY

	$x^2/a^2 + y^2/b^2 = 1$ **a > b**	$x^2/a^2 + y^2/b^2 = 1$ **a < b**
Coordinates of the centre	(0, 0)	(0, 0)
Coordinate of the vertices	(± a, 0)	(0, ± b)
Foci of the ellipse	(± ae, 0)	(0, ± be)
Length of the major axis	2a	2b
Length of the minor axis	2b	2a
Equation of Major axis	y = 0	x = 0
Equation of Minor axis	x = 0	y = 0
Equation of Directrices	x = ± a/e	y = ± b/e
Eccentricity	$e^2 = 1 - (b^2/a^2)$	$e^2 = 1 - (a^2/b^2)$
Length of the latus-rectum	$2b^2/a$	$2a^2 / b$
Focal distance of a point	a ± ex	b ± ey

PROPERTIES

1. The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.
2. The product of the two perpendicular distances from the foci on any tangent of a ellipse is equal to b^2.
3. The tangent and normal at a point P of an ellipse bisects the internal and external angles between the focal distances of the point.
4. If the tangent at a point P on an ellipse meets the directrix in Q the PQ subtends a right angle at the corresponding focus.
5. The tangents at the ends of a focal chord of an ellipse intersect on the corresponding directrix.

FOCAL DISTANCE OF A POINT

The sum of the focal distance of any point on the ellipse is equal to the major axis.

POSITION OF A POINT WITH RESPECT TO ELLIPSE

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow S = \frac{x^2}{a^2}+\frac{y^2}{b^2} - 1$

$S_1 = \frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2} - 1$

$\therefore$ The point (x_1, y_1) lies

- **Outside** the ellipse if $S_1 > 0$.
- **On** the ellipse if $S_1 = 0$
- **Inside** the ellipse if $S_1 < 0$

INTERSECTION OF LINE AND ELLIPSE

The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

- In **two real and distinct points** if $c^2 < a^2 m^2 + b^2$
- In **two imaginary points** if $c^2 > a^2 m^2 + b^2$
- In **two real and coincident points** i.e., the line touch the ellipse if $c^2 = a^2 m^2 + b^2$.

  $\therefore$ The equation of tangent is given by $y = mx \pm \sqrt{a^2m^2 + b^2}$

  $\therefore$ **Point of Contact** is $\left[\frac{-a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right]$

EQUATION OF TANGENT **($S_{11} = 0$)**

The equation of tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$

EQUATION OF NORMAL

The equation of tangent at (x_1, y_1) to the ellipse is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

Slope of the above tangent, $m = \frac{-b^2x_1}{a^2y_1}$

$\therefore$ Slope of normal = $\dfrac{a^2y_1}{b^2x_1}$.

$\therefore$ Slope of the required normal = –1/m = $\dfrac{-b^2x_1}{a^2y_1}$.

Hence the required equation of the normal is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1) \quad \text{or} \quad \frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}$$

NUMBER OF TANGENTS DRAWN FROM A POINT TO AN ELLIPSE

Two tangents can be drawn from a point to an ellipse.

The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

NUMBER OF NORMALS

Four normals can be drawn from a point to an ellipse.

Examples

Ex. The co-ordinates of the foci of the ellipse $9x^2 + 5y^2 - 30y = 0$ are:

(A) (0, 5) and (0, 1) (B) (0, 2) and (0, – 2)

(C) (2, 0) and (– 2, 0) (D) (5, 0) and (– 5, 0)

Sol. The given equation can be written as $9x^2 + 5(y - 3)^2 = 45$

or $\dfrac{x^2}{5} + \dfrac{(y-3)^2}{9} = 1$, or , $\dfrac{X^2}{5} + \dfrac{Y^2}{9} = 1$

where x = X, y – 3 = Y. Here $a^2 = 5$ and $b^2 = 9$. Because $b^2 > a^2$

$\therefore a^2 = b^2(1 - e^2)$

$\therefore 5 = 9(1 - e^2) \therefore e = 2/3$.

If S and S' are foci of the ellipse, then CS = CS' = be = 3. 2/3 = 2.

$\therefore$ Co – ordinates of focus X = 0, Y = ± 2 or, x = 0, y – 3 = ± 2

$\therefore$ The focus of ellipse are (0, 1) and (0, 5)

$\therefore$The correct answer is (a).

Ex. The equation of the ellipse whose focus is (– 1, 1), directrix is x – y + 4 = 0 and eccentricity is $\dfrac{1}{\sqrt{2}}$ is given by:

(A) $3x^2 + 2xy + 3y^2 = 8$ (B) $2x^2 + 2xy + 3y^2 = 8$

(C) $3x^2 + 2xy + 2y^2 = 8$ (D) $3x^2 + 2y^2 = 8$

Sol. The equation of the ellipse is $(x + 1)^2 + (y - 1)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2 \times \dfrac{(x - y + 4)^2}{(1)^2 + (-1)^2}$

or $4[x^2 + y^2 + 2x - 2y + 2] = (x^2 + y^2 - 2xy + 8y - 8y + 16)$

or $3x^2 + 3y^2 + 2xy - 8 = 0$.

$\therefore$The correct answer is (A).

Ex. The condition that the line lx + my + n = 0 touches the ellipse $ax^2 + by^2 = 1$ is:

(A) $n^2 = a^2 l^2 + b^2 m^2$ (B) $n^2 = \frac{l^2}{a} + \frac{m^2}{b}$ (C) $n^2 = \frac{l^2}{a^2} + \frac{m^2}{b^2}$ (D) none of these

Sol. The given ellipse is $\frac{x^2}{1/a} + \frac{x^2}{1/b}$ = 1. Also the given line can be written as $y = -\frac{1}{m}x - \frac{n}{m}$.

Since the condition of tangency is $c^2 = a^2 m^2 + b^2$

$\therefore \left(-\frac{n}{m}\right)^2 = \frac{1}{a}\left(-\frac{1}{m}\right)^2 + \frac{1}{b}$ or $n^2 = \frac{l^2}{a} + \frac{m^2}{b}$.

$\therefore$ The correct answer is (B).

Ex. Find the equation to the ellipse, whose centres are the origin, whose axes are the axes of co-ordinates, and which pass through

(a) the points (2, 2) and (3, 1) and

(b) the points (1, 4) and (– 6, 1).

Sol. Assume the equation of ellipse to be $x^2/a^2 + y^2/b^2 = 1$

Put the two points and solve the two simultaneous equations to get a and b.

Ans. (a) $3x^2 + 5y^2 = 32$, and (b) $3x^2 + 7y^2 = 115$.

Ex. Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

Sol. $2b^2/a = 10$ and 2b = 2ae,

Also $b^2 = a^2 (1 - e^2)$.

Ans $x^2 + 2y^2 = 100$.

Ex. Is the point (4, – 3) within or outside the ellipse $5x^2 + 7y^2 = 11$?

Sol. As the equation of the ellipse is $5x^2 + 7y^2 - 11 = 0$ ….(1) and the point is (4, – 3)

Putting the co-ordinates (4, – 3) in (1), we have

L.H.S = 80 + 63 – 11 = 132 which is positive.

Hence the point (4, – 3) lie outside the ellipse

Ex. Show that $x^2 + 4y^2 + 2x + 16y + 13 = 0$ is the equation of an ellipse. Find its eccentricity, vertices, foci, directrices, length of the latus-rectum and the equation of the latus-rectum.

Sol. We have $x^2 + 4y^2 + 2x + 16y + 13 = 0$

$\Rightarrow (x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$

$\Rightarrow (x + 1)^2 + 4(y + 2)^2 = 4 \Rightarrow \frac{(x+1)^2}{2^2} + \frac{(y+2)^2}{1^2} = 1$ … (i)

Shifting the origin at (– 1, – 2) without rotating the coordinate axes and denoting the new coordinate with respect to the new axes by X and Y, we have x = X – 1 and y = Y – 2. …(ii)

Using these relations, equation (i) reduces to $\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$ … (iii)

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2}$ = 1, where a = 2 and b = 1.

Thus, the given equation represents and ellipse.

Clearly a > b

So, the given equation represents an ellipse whose major and minor axes are along X and Y axes respectively.

Eccentricity: The eccentricity e is given by e = $\sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$.

Vertices: The vertices of the ellipse with respect to the new axes are (X = ± a, Y = 0),
i.e. (X = ± 2, Y = 0).
So, the vertices with respect to the old axes are given by (± 2 –1, – 2), i.e., (– 3, 2) and (1, –2) [Using (ii)]

Foci: The coordinates of the foci with respect to the new axes are given by (X = ± ae, Y = 0)
i.e., (X = ± √3, Y = 0). So, the coordinates of foci with respect to the old axes are given by (√3 – 1, – 2) [Putting (X = ± √3, Y = 0) in (ii)]

Directices: The equation of the directrices with respect to the new axes are X = ± a/e, i.e., X = ± 4/√3.

So, the equations of the directrices with respect to the old axes are x = ± $\frac{4}{\sqrt{3}}$ – 1 i.e.

x = $\frac{4}{\sqrt{3}}$ – 1 and x = –$\frac{4}{\sqrt{3}}$ – 1 [Putting X = ± $\frac{4}{\sqrt{3}}$ in (ii)]

Length of the Latus-rectum: The length of the latus-rectum = $2b^2/a = 2/2 = 1$.

Equations of Latus-rectum: The equation of the latus-rectum with respect to the new axes are
X = ± √3 – 1. [Putting X = ± √3 in (ii)],
i.e., x = √3 – 1 and x = –√3 – 1.

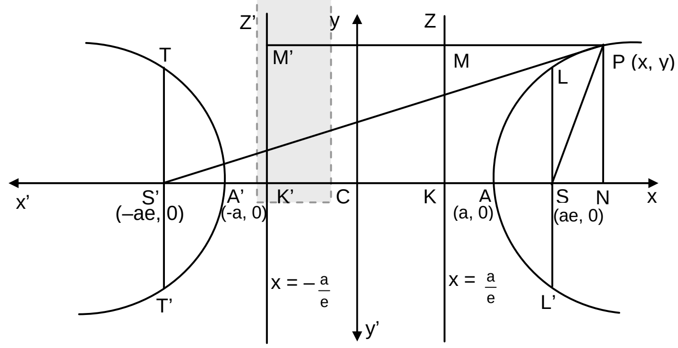
Ex. Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi-minor axis is √5.

Sol. Let S and S' be two foci of the required ellipse.
Then the coordinates of S and S ' are (2, 3) and (–2, 3) respectively.
Therefore SS' = 4.
Let 2a and 2b be the lengths of the axes of the ellipse and e be its eccentricity.
Then SS' = 2 ae. ⇒ 2ae = 4 ⇒ ae = 2.
Now, $b^2 = a^2 (1 - e^2)$
⇒ $5 = a^2 - 2^2$ ⇒ a = 3.
Let P (x, y) be any point on the ellipse. Then SP + S ' P = 2a
⇒ $\sqrt{(x-2)^2+(y-3)^2} + \sqrt{(x+2)^2+(y-3)^2} = 6$ ⇒ $5x^2 + 9y^2 + 72x - 54y + 36 = 0$.
This is the required equation of the ellipse.

THE HYPERBOLA

HYPERBOLA: The hyperbola is a locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point is called the focus and the fixed line is called the directrix. Hence, if S be the focus of the hyperbola, P any point on it and PN is perpendicular distance from P on the directrix, then $\frac{SP}{PM} = e \Rightarrow SP = e\,PM, (e > 1)$

For hyperbola, **eccentricity (e) > 1**



EQUATION OF HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is the standard equation of hyperbola, where $b^2 = a^2(e^2 – 1)$

GENERAL EQUATION OF HYPERBOLA

The equation of the hyperbola whose focus is the point (h, k), the directrix is the line Ax + By + C = 0 and whose eccentricity is e, is given by

$$(x – h)^2 + (y – k)^2 = e^2 \frac{(Ax + By + C)^2}{A^2 + B^2}$$

FOCAL DISTANCE OF A POINT

The difference of the focal distance of a point on the hyperbola is constant and equal to the transverse axis i.e. S' P – SP = 2a = the transverse axis AA'

COMPARISON TABLE		
	Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	**Conjugate hyperbola** $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the vertices	(a, 0) and (– a, 0)	(0, b) and (0, – b)
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	2a	2b
Length of the conjugate axis	2b	2a
Equation of the directories	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e^2 = 1+\frac{b^2}{a^2}$	$e^2 = 1+\frac{a^2}{b^2}$
Length of the latus rectum	$2b^2/a$	$2a^2/b$
Equation of the transverse axis	y = 0	x = 0
Equation of the conjugate axis	x = 0	y = 0

RESULTS FOR HYPERBOLA

Since the fundamental equation of the hyperbola differs from that of the ellipse only in having $-b^2$ instead of b^2, most of the results 'obtained for the ellipse hold good for hyperbola and there will be change of sign of b^2. The results are:

1. **Tangent** at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ **($S_{11}=0$)**

2. **Normal** at the point (x_1, y_1) is $\frac{x-x_1}{\frac{x_1}{a^2}}-\frac{y-y_1}{-\frac{y_1}{b^2}}$

3. **Condition of Tangency:** The line y = mx + c touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$

   if $\mathbf{c^2 = a^2 m^2 - b^2}$

   $\therefore$ Any tangent to the hyperbola is $y = mx \pm \sqrt{a^2m^2-b^2}$. And point of contact is given by

   $$\left[\frac{-a^2m}{\sqrt{a^2m^2-b^2}}, \frac{-b^2}{\sqrt{a^2m^2-b^2}}\right]$$

4. Chord of contact of the point (x_1, y_1) is : $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ **($S_{11}=0$)**

ASYMPTOTE

An Asymptote to a curve is a straight line which touches the curve at infinity, but which itself does not lie wholly at infinity.

Equations of Asymptotes of a Hyperbola:

$\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$

are the equations of a Hyperbola.

The combined equation of the Asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

RECTANGULAR HYPERBOLA

A hyperbola in which the semi-transverse and semi-conjugate axes are equal is called a **rectangular** or **equilateral** hyperbola.

Thus, in this case b = a i.e., coefficient of x^2 + coefficient of $y^2 = 0$.

$\therefore$**The equation of rectangular hyperbola with centre as origin and usual axes is**

$$x^2 - y^2 = a^2.$$

$\therefore$The **eccentricity** of rectangular hyperbola is **$\sqrt{2}$.**

Equation of rectangular hyperbola referred to its asymptotes as axes.

The equation of rectangular hyperbola whose asymptotes taken as axes is

$xy = \frac{1}{4}(a^2 + a^2) = a^2/2$ or, $xy = c^2$ writing $a^2 = 2c^2$

Note: When the equation of the rectangular hyperbola is $xy = c^2$ then

1. The asymptotes are the co-ordinate axes.
2. The axes are the bisectors of the angles between the co-ordinates and their equations are $y = \pm x$.

NUMBER OF TANGENTS DRAWN FROM A POINT TO A HYPERBOLA

Two tangents can be drawn from a point to a hyperbola.

NUMBER OF NORMALS FROM A POINT TO A HYPERBOLA

In general four normals can be drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Examples:

Ex. Find the eccentricity, directrix and co-ordinates of foci of the hyperbola $x^2 + 2x - y^2 + 5 = 0$

Sol. The given equation can be written as $(x + 1)^2 - y^2 = -4$

or, $\frac{(x+1)^2}{-4} + \frac{y^2}{4} = 1 \Rightarrow \frac{Y^2}{4} - \frac{X^2}{4} = 1$

where Y = y, X = x + 1. Here $a^2 = 4$, $b^2 = 4$.

$\therefore\ a^2 = b^2(e^2 - 1) \Rightarrow 4 = 4(e^2 - 1) \Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$.

Equation of directrix $y = \pm b/e = \pm 2/\sqrt{2}$.

The co-ordinate of foci $(-1, \pm be)$ i.e. $(-1, \pm 2\sqrt{2})$.

Ex. The locus of $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$ is a

(A) circle (B) parabola (C) ellipse (D) hyperbola

Sol. Here a = 3, b = 8, h = 5, g = 7, f = 11, c = 7

$abc + 2fgh - af^2 - bg^2 - ch^2 = 3 \times 8 \times 7 + 2 \times 11 \times 7 \times 5 - 3 \times (11)^2 - 8 \times (7)^2 - 7 \times (5)^2 \neq 0$

and $h^2 - ab = 25 - 24 = 1 > 0$

i.e. $h^2 > ab$

$\therefore$ The given locus is a hyperbola

$\therefore$The correct answer is (D).

Ex. Find the equation of the hyperbola length of whose conjugate axis is 7 and which passes through the point (3, – 2)

Sol. b = 7/2 and a can be found by substituting the given point in the standard equation.
$65x^2 - 36y^2 = 441$ is the required equation

Ex. Show that the equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represents a hyperbola. Find the coordinates of the centre, lengths of the axes, eccentricity, latus-rectum, coordinates of foci and vertices, equations of the directrices of the hyperbola.

Sol. We have $9x^2 - 16y^2 - 18x + 32y - 151 = 0 \Rightarrow 9(x^2 - 2x) - 16(y^2 - 2y) = 151$
$\Rightarrow (x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144 \Rightarrow 9(x - 1)^2 - 16(y - 1)^2 = 144$

$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1 \quad \dots \text{(i)}$$

Shifting the origin at (1, 1) without rotating the axis and denoting the new coordinates with respect to these axes by X and Y, we have x = X + 1 and y = Y + 1 ... (ii)

Using these relations, equations (i) reduces to $\frac{X^2}{16} - \frac{Y^2}{9} = 1$ ... (iii).

Centre: The coordinates of the centre with respect to the new axes are (X = 0, Y = 0)
So, the coordinates of the centre with respect to the old axes are (1, 1) [Putting X = 0, Y = 0 in (ii)]
Transverse Axis: Length of the transverse axis = 2a = 8
Conjugate Axis: Length of the conjugate axis = 2b = 6
Eccentricity: The eccentricity e = 5/4
Latus-rectum: Length of the latus-rectum = 9/2.
Foci: The coordinates of foci with respect to the new axes are (X = ± 5, Y = 0).i.e, (X = ± 5, Y = 0).
So, the coordinates of foci with respect to the old axes are (1 ± 5, 1), i.e., (6, 1) and (– 4, 1) [Putting X = ± 5, Y = 0 in (ii)]
Vertices: The coordinates of the vertices with respect to the new axes are (X ± a, Y = 0), i.e., (X = ±4, Y = 0).
So, the coordinates of the vertices with respect to the old axes are (±4 + 1, 1) i.e., (5, 1) and (–3, 1) [Putting X = ±4, Y = 0 in (ii)]
Directices: The equations of the directrices with respect to the new axes are X = ±a/e. i.e. X = ±16/5.
So, the equations of the directrices with respect to the old axes are x = 21/5 and x = –11/5.

Ex. Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:
(a) Vertices at (± 5, 0), Foci at (± 7, 0)
(b) Vertices at (0, ± 7), e = 4/3

Sol. (i) 2a = 10, 2ae = 14, which gives the equation as $x^2/25 - y^2/24 = 1$.
(ii) 2b = 14, e = 4/3, which gives the equation as $9x^2/343 - y^2/49 = 1$.

Ex. Find the equation of the hyperbola whose foci are (8, 3), (0, 3), and eccentricity = 4/3.

Sol. The centre is (4, 3) and 2ae = 8, which gives a = 3 and b = √7.

Hence the equation is $\frac{(x-4)^2}{9} - \frac{(x-3)^2}{7} = 1$.

Ex. For what value of λ does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$.

Sol. Using $c^2 = a^2 m^2 - b^2$, we get, $\lambda = \pm 6$.

Ex. Find the equations of the tangents and normal to the hyperbola $16x^2 - 9y^2 = 144$ at (5, 8/3).

Sol. Tangent is $10x - 3y = 18$

Also, normal is a line perpendicular to tangent and passing through the same point.

Hence we've to find the equation of line having slope –3/10 and passing through (5, 8/3).

The equation will be $9x + 30y = 125$.